

Shadows and convexity of surfaces

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Abstract

We study the geometry and topology of immersed surfaces in Euclidean 3-space whose Gauss map satisfies a certain two-piece-property, and solve the “shadow problem” formulated by H. Wente.

1. Introduction

Let M be a closed oriented 2-dimensional manifold, $f: M \rightarrow \mathbb{R}^3$ be a smooth immersion into Euclidean 3-space, and $n: M \rightarrow \mathbb{S}^2$ be a unit normal vectorfield, or the Gauss map, induced by f . Then for every unit vector $u \in \mathbb{S}^2$ (corresponding to the direction of light) the *shadow*, S_u , is defined by

$$S_u := \{p \in M : \langle n(p), u \rangle > 0\},$$

where $\langle \cdot, \cdot \rangle$ is the standard innerproduct. If f is a *convex embedding*, i.e., f maps M homeomorphically to the boundary of a convex body, then it is intuitively clear that S_u is a connected subset of M for each u . In 1978, motivated by problems concerning the stability of constant mean curvature surfaces, H. Wente [17] appears to have been the first person to study the converse of this phenomenon, which has since become known as the “shadow problem” [13]: *Does connectedness of the shadows S_u imply that f is a convex embedding?* In this paper we prove:

THEOREM 1.1. *f is a convex embedding if and only if, for every $u \in \mathbb{S}^2$, S_u is simply connected.*

Furthermore we show that the additional condition implied by the word “simply” in the above theorem is necessary:

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THEOREM 1.2. *There exists a smooth embedding of the torus, $f: \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$, such that for all $u \in \mathbb{S}^2$, S_u is connected.*

Thus, connectedness of the shadows in general is not strong enough to ensure convexity or even determine the topology; however, we can show:

THEOREM 1.3. *If M is topologically a sphere, and, for every $u \in \mathbb{S}^2$, S_u is connected, then f must be a convex embedding.*

In short, the answer to the above question is yes, provided that either the shadows are *simply* connected, or M is a sphere; otherwise, the answer is no. This settles Wente's shadow problem in 3-space. See [7] and [5] for motivations behind this problem and relations to constant mean curvature surfaces.

Note 1.4. The immersion $f: M \rightarrow \mathbb{S}^2$ has connected shadows if and only if for every great circle $C \subset \mathbb{S}^2$, $n^{-1}(\mathbb{S}^2 - C)$ has exactly two components. That is, the Gauss map satisfies a *two-piece-property* [3] similar to that formulated by T. Banchoff [2], and further developed by N. Kuiper [12].

Note 1.5. For a great circle $C \subset \mathbb{S}^2$, the number of components of $n^{-1}(\mathbb{S}^2 - C)$ has been called the *vision number* with respect to a direction perpendicular to C . This terminology is due to J. Choe, who conjectured [5, p. 210] that there always exists a direction with respect to which the vision number of $f: M \rightarrow \mathbb{R}^3$ is greater than or equal to $4 - \chi(M)$ where χ is the Euler characteristic. Theorem 1.2 gives a counterexample to this conjecture.

2. Regularity of horizons and shadow boundaries

First we need to establish some basic regularity results regarding the generic behavior of shadows. For each $u \in \mathbb{S}^2$, define the *shadow function* $\sigma_u: M \rightarrow \mathbb{R}$ by

$$\sigma_u(p) := \langle n(p), u \rangle.$$

$H_u := \sigma_u^{-1}(0)$ is called the *horizon* [5] in the direction u . It is easy to see that in general $\partial S_u \neq H_u \neq \partial S_{-u}$, where ∂ denotes the boundary; however, using Sard's theorem, we can show

PROPOSITION 2.1. *For almost all $u \in \mathbb{S}^2$ (in the sense of Lebesgue measure) H_u is a regular curve. Thus for these u , both ∂S_u and ∂S_{-u} are regular curves as well. Further, if H_u is connected, then $\partial S_u = H_u = \partial S_{-u}$.*

We say that $\Gamma \subset M$ is a *regular curve* if for each $p \in \Gamma$ there is an open neighborhood U of p in M and a homeomorphism $\varphi: U \rightarrow \mathbb{R}^2$ such that $\varphi(U \cap \Gamma) = \mathbb{R}$. In particular, unless stated otherwise, a regular curve needs not be differentiable.

Proof. Let $T_p M$ be the tangent plane of M at p which we identify with a subspace of \mathbb{R}^3 (by identifying $T_p M$ with $f_*(T_p M)$, and parallel translating the elements of $f_*(T_p M)$ to the origin in \mathbb{R}^3 ; f_* denotes the *differential* of f). Let $UTM := \{(p, u) : p \in M, u \in T_p M, \|u\| = 1\}$ denote the unit tangent bundle of M , and τ be the mapping given by

$$UTM \ni (p, u) \xrightarrow{\tau} u \in \mathbb{S}^2.$$

By Sard's theorem almost every $u \in \mathbb{S}^2$ is a regular value of τ ; consequently, for such u , $\tau^{-1}(u)$ is a regular curve in UTM .

Now let π be the mapping defined by

$$UTM \ni (p, u) \xrightarrow{\pi} p \in M,$$

and let u be a regular value of τ . Note that π is injective on $\tau^{-1}(u)$. As $\tau^{-1}(u)$ is compact, this implies that $\pi: \tau^{-1}(u) \rightarrow M$ is an embedding. Further note that

$$\pi(\tau^{-1}(u)) = \{p \in M : u \in T_p M\} = \{p \in M : \langle n(p), u \rangle = 0\} = H_u.$$

Thus H_u is a regular curve. But then, it follows that ∂S_u and ∂S_{-u} are each open in H_u , which yields that ∂S_u and ∂S_{-u} are both regular curves as well. Finally, since these shadow boundaries are also closed in H_u , it follows that whenever H_u is connected we have $\partial S_u = H_u = \partial S_{-u}$. \square

Note 2.2. Suppose that there is an open set $U \subset \mathbb{S}^2$, such that, for all $u \in U$, both S_u and S_{-u} are simply connected. Then M is homeomorphic to \mathbb{S}^2 ; because, by the above proposition, there exists a $u_0 \in U$ such that H_{u_0} is a regular curve. Consequently the closures $\overline{S_{u_0}}$ and $\overline{S_{-u_0}}$ are homeomorphic to disks. Further, since by assumption $M - H_{u_0}$ is made up of a pair of simply connected components, H_{u_0} is connected. Thus by the above proposition $\partial S_{-u_0} = \partial S_{u_0}$. So M is homeomorphic to a pair of disks glued together along their boundaries.

By *smooth* we mean differentiable of class C^∞ , and for convenience we always assume that the immersion $f: M \rightarrow \mathbb{R}^3$ is smooth, though in this paper it is enough that f be C^3 .

Note 2.3. The embedding $\pi: \tau^{-1}(u) \rightarrow M$ in the above proposition is smooth, when u is a regular value of τ . In particular, H_u is smooth for almost all $u \in \mathbb{S}^2$. To see this let $(p, u) \in \tau^{-1}(u)$. Then $u \in T_p M$. Let $v \in T_p M$ with $\langle u, v \rangle = 0$. Then $c(t) := (p, \cos(t)u + \sin(t)v)$ parametrizes the fiber $UT_p M$ of the unit tangent bundle. Note that

$$\tau_{*(p,u)}(c'(0)) = \frac{d}{dt} \tau(p, \cos(t)u + \sin(t)v) \Big|_{t=0} = v \neq 0.$$

On the other hand,

$$T_{(p,u)}(\tau^{-1}(u)) = \left\{ X \in T_{(p,u)}(UTM) : \tau_{*(p,u)}(X) = 0 \right\}.$$

Thus $c'(0) \notin T_{(p,u)}(\tau^{-1}(u))$, which implies that $\tau^{-1}(u)$ is never tangent to any of the fibers $UT_p M$ of the unit tangent bundle. So $\pi|_{\tau^{-1}(u)}$ is a smooth immersion.

Next we need a local regularity result for the horizons and shadow boundaries. The *Gaussian curvature* $K: M \rightarrow \mathbb{R}$ is defined by $K(p) := \det(n_*(p))$.

PROPOSITION 2.4. *If $K(p) \neq 0$ for some $p \in M$, then there exists a neighborhood U of p such that for all $u \in T_p M$, $H_u \cap U$ is a smooth regular curve and $\partial S_u \cap U = H_u \cap U = \partial S_{-u} \cap U$.*

Proof. Since $\det(n_{*p}) = K(p) \neq 0$, then, by the inverse function theorem, n is a diffeomorphism between small neighborhoods U of p in M and V of $n(p)$ in \mathbb{S}^2 . Let $\mathbb{S}_u^2 := \{x \in \mathbb{S}^2 : \langle x, u \rangle > 0\}$. Then $\partial \mathbb{S}_u^2 = \partial \mathbb{S}_{-u}^2$ is a regular curve. Thus, since $S_u = n^{-1}(\mathbb{S}_u^2)$ and $S_{-u} = n^{-1}(\mathbb{S}_{-u}^2)$, the proof follows. \square

Note 2.5. If $K(p) = 0$, then H_u may not be regular for all $u \in T_p M$; however, typically H_u will be regular for most $u \in T_p M$; because, for $u \in T_p M$, the differential of σ_u at p is given by

$$(d\sigma_u)_p(\cdot) = \langle \cdot, n_{*p}(u) \rangle.$$

So if $n_{*p}(u) \neq 0$, e.g., u is not an *asymptotic direction*, then $d\sigma_u$ is nonzero at p . Consequently, by the implicit function theorem, $\sigma_u^{-1}(\sigma_u(p)) = \sigma_u^{-1}(0) = H_u$ is a smooth regular curve near p .

3. Critical points of height functions

The next set of preliminary results we need involves some basic applications of Morse theory [14]. For every $u \in \mathbb{S}^2$, let the *height function* $h_u: M \rightarrow \mathbb{R}$, associated to the immersion $f: M \rightarrow \mathbb{R}^3$, be defined by

$$h_u(p) := \langle f(p), u \rangle.$$

Recall that p is a *critical point* of h_u if the differential map $(dh_u)_p: T_p M \rightarrow \mathbb{R}$ is zero. Since $(dh_u)_p(\cdot) = \langle \cdot, u \rangle$, it follows that p is a critical point of h_u if and only if $u = \pm n(p)$. If all of its critical points are nondegenerate, h_u is a *Morse function*.

LEMMA 3.1. (i) h_u is a Morse function if and only if $K \neq 0$ at all critical points of h_u . (ii) h_u is a Morse function for almost all $u \in \mathbb{S}^2$. (iii) The set $U \subset \mathbb{S}^2$ such that for all $u \in U$ h_u is a Morse function is open.

Though the above is fairly well-known (e.g., see [3, pp. 11–12]), we include a brief proof for completeness.

Proof. If p is a critical point of h_u , then, as a standard computation shows, the Hessian of h_u is given by

$$\text{Hess}h_u(\cdot, \cdot) = \pm \langle \cdot, n_{*p}(\cdot) \rangle.$$

Thus h_u is a Morse function if and only if at each critical point p , $K(p) = \det(n_{*p}) \neq 0$. This is equivalent to requiring that both u and $-u$ be regular values of n , because p is a critical point of h_u if and only if $u = \pm n(p)$. Let $U \subset \mathbb{S}^2$ be the set of all such values. Then, by Sard's theorem, $\mathbb{S}^2 - U$ has measure zero. Further, since M is compact, and the set of critical points of n is closed, it follows that the set of critical values of n is closed as well, so U is open. \square

The following is implicit in a paper of Chern and Lashof [4].

LEMMA 3.2. *If f is not a convex embedding, then there exists a Morse height function h_u with at least three critical points.* \square

Proof. Let $\#C(h_u)$ denote the number of critical points of h_u . Since p is a critical point of h_u if and only if $n(p) = \pm u$, we have:

$$\int_{\mathbb{S}^2} \#C(h_u) du = \int_{\mathbb{S}^2} \#n^{-1}(\pm u) du = 2 \int_M |\det(n_*)| dV = 2 \int_M |K| dV.$$

The second equality above is just an application of the area formula [6, Thm. 3.2.3], where dV denotes the volume element on M . Suppose that f is not a convex embedding. Then, by a well-known theorem of Chern and Lashof [4],

$$\int_M |K| dV > 4\pi.$$

Combining the above expressions yields a lower bound for the average number of critical points:

$$\frac{1}{4\pi} \int_{\mathbb{S}^2} \#C(h_u) du > 2.$$

So since, by Lemma 3.1, h_u is a Morse function for almost every $u \in \mathbb{S}^2$, it follows that there exists a Morse function such that $\#C(h_u) > 2$. \square

4. Triplets on the boundaries of simply connected domains

Here we develop some elementary topological methods whose motivation will become more clear in the next section.

Definition 4.1. By a *domain* we mean a connected open subset $\Omega \subset M$. We say Ω is *adjacent* to a triplet of points $\{p_1, p_2, p_3\} \subset M$ if $p_i \in \partial\Omega$. Ω is *regular* near p_i if there are open neighborhoods U_i of p_i and homeomorphisms $\varphi_i: U_i \rightarrow \mathbb{R}^2$ which map $U_i \cap \Omega$ into the upper half-plane. A simple closed curve $T \subset \overline{\Omega}$ is a *triangle* of Ω (with vertices at $\{p_1, p_2, p_3\}$) if $p_i \in T$, and $T - \{p_1, p_2, p_3\} \subset \Omega$.

The following lemma, though quite elementary, is more subtle than it might at first appear (see Note 4.3).

LEMMA 4.2. *Every domain Ω adjacent to $\{p_1, p_2, p_3\}$ admits a triangle. Further if Ω is simply connected and regular near p_i , then any pair of such triangles may be homotoped to each other through a family of triangles of Ω .*

Proof. Since Ω is open and connected, there exists a regular arc $A_{12} \subset \Omega$ whose end points are p_1 and p_2 . Since A_{12} is regular, there exists a component $(\Omega - A_{12})^+$ of $\Omega - A_{12}$ which contains p_3 in its closure. Let $A_{23} \subset (\Omega - A_{12})^+$ be a regular arc with end points on p_2 and p_3 . Then, similarly, there exists a component $((\Omega - A_{12})^+ - A_{23})^+$ of $(\Omega - A_{12})^+ - A_{23}$ which contains p_1 in its closure. Finally, let $A_{31} \subset ((\Omega - A_{12})^+ - A_{23})^+$ be a regular arc with end points at p_3 and p_1 . The union of these three arcs, and their endpoints, gives the desired triangle.

Now suppose that Ω is simply connected and regular near p_i . Let T and T' be a pair of triangles of Ω , and let A_{12} and A'_{12} be arcs of T and T' respectively which connect p_1 and p_2 . Since Ω is regular near p_i , we may homotope A_{12} (while keeping its end points fixed) by a small perturbation near p_1 so that A_{12} and A'_{12} coincide along a segment near p_1 . Similarly, we may assume that they coincide near p_2 as well. Then it remains to homotope proper subarcs of A_{12} and A'_{12} which coincide at a pair of end points in Ω . Since Ω is simply connected, these subarcs may be homotoped to each other while keeping the end points fixed. Thus A_{12} and A'_{12} are homotopic through a family of arcs of Ω with end points at p_1 and p_2 . Other arcs of T may be similarly homotoped to their counterparts in T' , which completes the proof. \square

Note 4.3. Without the regularity assumption near p_i , the second claim in the above lemma is not true in general: Suppose for instance that $\Omega \subset \mathbb{R}^2$ is an open disk of radius 1 centered at the origin, and with segment $[0, 1)$ removed. Set $p_1 = (0, 0)$, $p_2 = (1/2, 0)$, and $p_3 = (1, 0)$. Then a triangle of Ω which lies above the x -axis may not be homotoped to one lying below the x -axis.

PROPOSITION 4.4. *For a fixed orientation of M , every simply connected domain Ω which is adjacent to and regular near a triple of (distinct) points $\{p_1, p_2, p_3\} \subset M$ uniquely determines a permutation α_Ω of $\{p_1, p_2, p_3\}$ such that (i) if Ω and Ω' have a triangle in common, then $\alpha_\Omega = \alpha_{\Omega'}$; and (ii) if $\partial\Omega = \partial\Omega'$ is a regular curve, and Ω and Ω' are distinct, then $\alpha_\Omega \neq \alpha_{\Omega'}$.*

Proof. By Lemma 4.2 there exists a triangle T of Ω . T bounds a simply connected subdomain U of Ω . Since M is oriented, U inherits a preferred sense of orientation, which in turn induces an orientation, or a sense of direction, on T . This direction induces a permutation of $\{p_1, p_2, p_3\}$ in the obvious way: If as we move along T and pass p_1 we reach p_2 before reaching p_3 , then we set the induced permutation α_Ω to be the cycle (p_1, p_2, p_3) ; otherwise, the induced permutation is the cycle (p_1, p_3, p_2) . It is clear that these permutations depend continuously on T . Thus, since by Lemma 4.2, all triangles of Ω are homotopic, it follows that α_Ω does not depend on the choice of T and is therefore well defined; and furthermore, if Ω and Ω' have a triangle in common then $\alpha_\Omega = \alpha_{\Omega'}$.

Now suppose that $\partial\Omega = \partial\Omega'$ is a regular curve, and Ω and Ω' are distinct. Then Ω and Ω' induce opposite orientations on $\partial\Omega$ which in turn gives rise to distinct permutations of $\{p_1, p_2, p_3\}$ (since Ω is simply connected, $\partial\Omega$ is connected). But by small perturbations, $\partial\Omega$ may be homotoped to a triangle of Ω , just as well as it may be homotoped to a triangle of Ω' . Thus the orientations which Ω and Ω' induce on $\partial\Omega$ are consistent with the orientations which Ω and Ω' induce on their own triangles respectively. So $\alpha_\Omega \neq \alpha_{\Omega'}$. \square

5. Proof of Theorem 1.1

First we show that if f is a convex embedding, then S_u is simply connected for all $u \in \mathbb{S}^2$. To see this let Π be a plane perpendicular to u and let $\pi: \mathbb{R}^3 \rightarrow \Pi$ be the orthogonal projection. Then $D := \pi(f(M))$ is a convex subset of Π with interior points. In particular, $\text{int}(D)$ is homeomorphic to an open disk. Since $f(M)$ is convex and by definition $\langle n(p), u \rangle > 0$ for all $p \in S_u$, it is not hard to verify that $f(S_u)$ is a graph over $\text{int}(D)$. Thus $\pi \circ f: S_u \rightarrow \text{int}(D)$ is a homeomorphism.

Now we prove the other direction: Assume that for every $u \in \mathbb{S}^2$, S_u is simply connected; we have to show that f is a convex embedding. The proof is by contradiction:

LEMMA 5.1. *If f is not a convex embedding, then there exists a pair of orthogonal vectors $u_0, v_0 \in \mathbb{S}^2$ such that (i) h_{u_0} is a Morse function with at least three critical points, and (ii) $\partial S_{v_0} = H_{v_0} = \partial S_{-v_0}$ is a regular curve.*

Proof. By Lemma 3.2, there exists a unit vector $u \in \mathbb{S}^2$ such that the corresponding height function h_u is a Morse function and has at least three critical points. Further, it follows from Lemma 3.1, that this u may be chosen from an open set $U \subset \mathbb{S}^2$.

Let $u^\perp := \{v \in \mathbb{S}^2 : \langle u, v \rangle = 0\}$. Then $U^\perp := \cup_{u \in U} u^\perp$ is open. Consequently, by Proposition 2.1, there exists a $v_0 \in u_0^\perp \subset U^\perp$ such that H_{v_0} is a regular curve. Further, since the complement of H_{v_0} consists of a pair of simply connected domains, H_{v_0} is connected. Thus, again by Proposition 2.1, $\partial S_{v_0} = H_{v_0} = \partial S_{-v_0}$ is a regular curve. \square

Let $\hat{v}_0 \in \mathbb{S}^2$ be a vector orthogonal to both u_0 and v_0 , and set

$$(1) \quad v(\theta) := \cos(\theta) v_0 + \sin(\theta) \hat{v}_0.$$

Let p_i , $i = 1, 2, 3$, be a fixed triple of (distinct) critical points of h_{u_0} .

LEMMA 5.2. *For all $\theta \in \mathbb{R}$, $S_{v(\theta)}$ is a domain adjacent to and regular near p_i .*

Proof. If p_i is a critical point of h_{u_0} , then $n(p_i) = \pm u_0$. So $\sigma_{v(\theta)}(p_i) = \langle v(\theta), \pm u_0 \rangle = 0$, which yields that $p_i \in H_{v(\theta)}$. Since h_{u_0} is a Morse function, then, by Lemma 3.1, $K(p_i) \neq 0$. So by Proposition 2.4, there exists a neighborhood U_i of p_i such that $\partial S_{v(\theta)} \cap U_i = H_{v(\theta)} \cap U_i = \partial S_{-v(\theta)} \cap U_i$, which completes the proof. \square

It now follows from Proposition 4.4 that each $S_{v(\theta)}$ induces a permutation of $\{p_1, p_2, p_3\}$ which we denote by $\alpha_\theta := \alpha_{(S_{v(\theta)})}$. Further, by the same proposition and since $\partial S_{v_0} = \partial S_{-v_0}$ is a regular curve, it follows that $\alpha_0 \neq \alpha_\pi$. On the other hand, letting Sym denote the symmetric group, we claim that the mapping

$$\mathbb{R} \ni \theta \longmapsto \alpha_\theta \in \text{Sym}(\{p_1, p_2, p_3\})$$

is locally constant, which, since $[0, \pi]$ is connected, would imply that $\alpha_0 = \alpha_\pi$. This contradiction, which would complete the proof, follows from Proposition 4.4 and the following:

LEMMA 5.3. *For each $\theta_0 \in \mathbb{R}$ there exists an $\varepsilon > 0$ such that if $|\theta - \theta_0| < \varepsilon$ then $S_{v(\theta)}$ and $S_{v(\theta_0)}$ have a common triangle (with vertices at $\{p_1, p_2, p_3\}$).*

Proof. Recall that, since h_{u_0} is a Morse function, then, by Lemma 3.1, $K(p_i) \neq 0$ which yields that n is a local diffeomorphism at p_i . Therefore, by Proposition 2.4, in a neighborhood W of $\{p_1, p_2, p_3\}$, $\partial S_{v(\theta)} = H_{v(\theta)} = n^{-1}(v^\perp(\theta))$ where $v^\perp(\theta)$ denotes the great circle in \mathbb{S}^2 orthogonal to $v(\theta)$. So, since $v^\perp(\theta)$ depends continuously on θ , it follows that, in W , $\partial S_{v(\theta)}$ depends continuously on θ as well.

Let T be a triangle of $S_{v(\theta_0)}$. Since $S_{v(\theta_0)}$ is open, after a perturbation of T we may assume that the arcs of T are smooth and meet $\partial S_{v(\theta_0)}$ transversely (recall that, by Proposition 2.4, $\partial S_{v(\theta_0)}$ is smooth near p_i). Thus, by the above paragraph, it follows that if $|\theta - \theta_0| < \varepsilon_1$, for some sufficiently small $\varepsilon_1 > 0$, then T meets $\partial S_{v(\theta)}$ transversely as well. Then it follows that for some neighborhood W of $\{p_1, p_2, p_3\}$, $(T - \{p_1, p_2, p_3\}) \cap W \subset S_{v(\theta)}$ for all θ such that $|\theta - \theta_0| < \varepsilon_1$.

Next note that $T - W$ is compact, and the mapping $\theta \mapsto \sigma_{v(\theta)}$ is continuous; therefore, since by assumption $\sigma_{v(\theta_0)} > 0$ on $T - W$, it follows that there exists an $\varepsilon_2 > 0$ such that $\sigma_{v(\theta)} > 0$ on $T - W$ for all θ such that $|\theta - \theta_0| < \varepsilon_2$. This yields that $T - W \subset S_{v(\theta)}$ for all θ such that $|\theta - \theta_0| < \varepsilon_2$.

From the previous two paragraphs it follows that, setting $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$, we have $(T - \{p_1, p_2, p_3\}) \subset S_{v(\theta)}$ for all θ such that $|\theta - \theta_0| < \varepsilon$, which completes the proof. \square

Note 5.4. Theorem 1.1 does not remain valid if the shadows are defined as the sets where $\langle n(p), u \rangle \geq 0$. For instance, the standard torus of revolution would be a counterexample.

Note 5.5. Theorem 1.1 does not remain valid without the compactness assumption; the hyperbolic paraboloid given by the graph of $z = xy$ would be a counterexample. This follows because here the unit normal vectorfield n is a homeomorphism into a hemisphere. Thus the preimage of any open hemisphere under n is simply connected.

6. Proof of Theorem 1.2

Definition 6.1. We say an immersion $\gamma: \mathbb{S}^1 \simeq \mathbb{R}/2\pi \rightarrow \mathbb{R}^3$ is a *skew loop* if it has no pair of distinct parallel tangent lines; i.e.,

$$\gamma'(t) \times \gamma'(s) \neq 0$$

for all $t, s \in [0, 2\pi)$, $t \neq s$.

A specific example of a skew loop, formulated by Ralph Howard [11], is as follows:

Example 6.2. Let $\gamma(t) := (x(t), y(t), z(t))$, where

$$\begin{aligned} x(t) &:= -\cos(t) - \frac{1}{20}\cos(4t) + \frac{1}{10}\cos(2t), \\ y(t) &:= +\sin(t) + \frac{1}{10}\sin(2t) + \frac{1}{20}\sin(4t), \\ z(t) &:= -\frac{46}{75}\sin(3t) - \frac{2}{15}\cos(3t)\sin(3t), \end{aligned}$$

and $t \in [0, 2\pi]$. A computation of the tangential indicatrix $T(t) := \gamma'(t)/\|\gamma'(t)\|$ shows that $T(t) \neq \pm T(s)$ for all $t, s \in [0, 2\pi)$, $t \neq s$. Thus γ is a skew loop. Figure 1 shows the pictures of a tube built around $\gamma(\mathbb{S}^1)$.

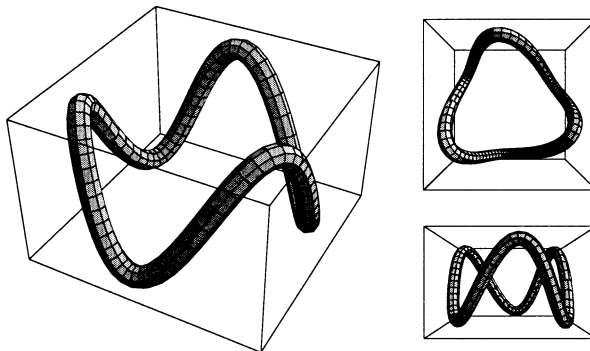


Figure 1

If $\gamma: \mathbb{S}^1 \rightarrow \mathbb{R}^3$ is an immersion, then the unit normal bundle of γ consists of all pairs $(p, \nu) \in \mathbb{S}^1 \times \mathbb{S}^2$ such that $\langle \gamma'(p), \nu \rangle = 0$. Since this bundle is homeomorphic to a torus, the following proposition yields Theorem 1.2.

PROPOSITION 6.3. *Let $\gamma: \mathbb{S}^1 \rightarrow \mathbb{R}^3$ be a skew loop and M be the unit normal bundle of γ . For $\varepsilon > 0$, define $f_\varepsilon: M \rightarrow \mathbb{R}^3$ by*

$$f_\varepsilon(p, \nu) := \gamma(p) + \varepsilon \nu.$$

Then, for ε sufficiently small, f_ε is a smooth immersion, and for all $u \in \mathbb{S}^2$, S_u is connected. If γ is an embedding, then f_ε is an embedding as well.

Proof. That f_ε is a smooth immersion and is an embedding when γ is embedded follows from the tubular neighborhood theorem. Let $n: M \rightarrow \mathbb{S}^2$ be the unit normal vector field given by $n(p, \nu) = \nu$, and $\pi: M \rightarrow \mathbb{S}^1$ be given by $\pi(p, \nu) = p$. For every $p \in \mathbb{S}^1$, let $F_p := \pi^{-1}(p)$ be the corresponding fiber. Note that n embeds F_p into the great circle in \mathbb{S}^2 which lies in the plane perpendicular to $T(p)$. Further recall that $S_u = n^{-1}(\mathbb{S}_u^2)$ where \mathbb{S}_u^2 is the open hemisphere determined by u . Thus there are only two possibilities for each $p \in \mathbb{S}^1$: either F_p intersects S_u in an open half-circle, or F_p is disjoint from S_u . The latter occurs if and only if $T(p)$ is parallel to u , which, since γ is skew, can occur at most once. Hence, it follows that S_u is either homeomorphic to a disk or an annulus. In particular, S_u is connected for every $u \in \mathbb{S}^2$. \square

Question 6.4. Let M be a closed oriented 2-dimensional manifold with topological genus $g(M) \geq 2$. Does there exist an embedding, or an immersion, $f: M \rightarrow \mathbb{R}^3$ such that S_u is connected for all $u \in \mathbb{S}^2$?

Note 6.5. Skew loops were first discovered by B. Segre [16] to disprove a conjecture of H. Steinhaus (see also [15]). More recently, it has been shown that there exists a skew loop in each knot class [18], and every pair of knots may be realized with the same tangential indicatrix [1].

Note 6.6. A general procedure for constructing skew loops is as follows. Let $T \subset \mathbb{S}^2$ be a smooth simple closed curve such that (i) the origin is contained in the interior of the convex hull of T , $(0, 0, 0) \in \text{int conv } T$, and (ii) T does not contain any pair of antipodal points, $T \cap -T = \emptyset$. Figure 2 shows an example.

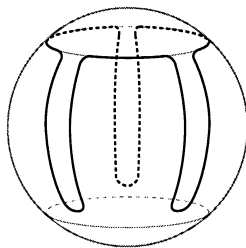


Figure 2

Let $T(s)$, $s \in \mathbb{R}$, denote a periodic parametrization of T by arclength. So, assuming T has total length L , we have $T(s + L) = T(s)$. Since $(0, 0, 0) \in \text{int conv } T$, there exists a function $\rho(s)$ with period L such that $\int_0^L \rho(s)T(s) ds = 0$ [10, p. 168]. Set

$$\gamma(t) := \int_0^t \rho(s)T(s) ds.$$

Then $\gamma(t + L) = \gamma(t)$. Further, $\gamma'(t)/\|\gamma'(t)\| = T(t)$. Thus γ is a closed curve whose tangential spherical image coincides with T . Hence γ is a skew loop.

Note 6.7. With the sole exception of ellipsoids, every closed surface immersed in \mathbb{R}^3 admits a skew loop [8].

7. Proof of Theorem 1.3

We follow a modified outline of the proof of Theorem 1.1, which again proceeds by contradiction. Suppose that M is homeomorphic to \mathbb{S}^2 and S_u is connected for all $u \in \mathbb{S}^2$. If f is not a convex embedding, let u_0 and v_0 be as in Lemma 5.1, and $v(\theta)$ be as defined by (1).

Definition 7.1. The *augmented shadow* $\tilde{S}_{v(\theta)}$ is the union of $S_{v(\theta)}$ with all components X of $H_{v(\theta)}$ such that $U - X \subset S_{v(\theta)}$ for an open neighborhood U of X .

Then $\tilde{S}_{v(\theta)}$ satisfies the conditions of the following lemma:

LEMMA 7.2. *If $U \subset \mathbb{S}^2$ is a connected open set, and $\mathbb{S}^2 - U$ is also connected and has an interior point, then U is simply connected.*

Proof. Let p be an interior point of $\mathbb{S}^2 - U$. Then the stereographic projection maps U into a connected open set with connected complement. Thus, by [9, Thm. 11.4.1], U is simply connected. \square

So $\tilde{S}_{v(\theta)}$ is simply connected. Further:

LEMMA 7.3. *For all $\theta \in \mathbb{R}$, $\tilde{S}_{v(\theta)}$ is a domain adjacent to and regular near p_i .*

Proof. This follows just as in the proof of Lemma 5.2, once we observe that whenever $\partial S_{v(\theta)} = H_{v(\theta)} = \partial S_{-v(\theta)}$ is regular in some open neighborhood, then $\partial \tilde{S}_{v(\theta)}$, and $\partial S_{v(\theta)}$ coincide within that neighborhood. \square

Thus each θ induces a permutation $\tilde{\alpha}_\theta := \alpha_{\tilde{S}_{v(\theta)}}^{\sim}$ of $\{p_1, p_2, p_3\}$ which satisfies the enumerated properties in Proposition 4.4. In particular $\tilde{\alpha}_0 \neq \tilde{\alpha}_\pi$, because since $\partial S_{v(0)} = \partial S_{-v(0)}$ is by Lemma 5.1 a regular curve, it follows that $\partial \tilde{S}_{v(0)} = \partial \tilde{S}_{-v(0)}$ is a regular curve as well. So it remains to verify the following lemma which shows that $\theta \mapsto \tilde{\alpha}_\theta$ is locally constant. This would yield that $\tilde{\alpha}_0 = \tilde{\alpha}_\pi$ which is the desired contradiction.

LEMMA 7.4. *For each $\theta_0 \in \mathbb{R}$ there exists an $\varepsilon > 0$ such that if $|\theta - \theta_0| < \varepsilon$ then $\tilde{S}_{v(\theta)}$ and $\tilde{S}_{v(\theta_0)}$ have a common triangle (with vertices at $\{p_1, p_2, p_3\}$).*

Proof. This is an immediate consequence of Lemma 5.3 where it was proved that $S_{v(\theta)}$ and $S_{v(\theta_0)}$ have a triangle in common (the proof of Lemma 5.3 makes no use of the simply connectedness assumption on $S_{v(\theta)}$). \square

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REFERENCES

- [1] C. ADAMS, C. LEFEVER, J. OTHMER, S. PAHK, A. STIER, and J. TRIPP, An introduction to the supercrossing index of knots and the crossing map, preprint.
- [2] T. F. BANCHOFF, The two-piece property and tight n -manifolds-with-boundary in E^n , *Trans. Amer. Math. Soc.* **161** (1971), 259–267.
- [3] T. E. CECIL and P. J. RYAN, *Tight and Taut Immersions of Manifolds*, Pitman (Advanced Publishing Program), Boston, Mass., 1985.
- [4] S.-S. CHERN and R. K. LASHOF, On the total curvature of immersed manifolds, *Amer. J. Math.* **79** (1957), 306–318.
- [5] J. CHOE, Index, vision number and stability of complete minimal surfaces, *Arch. Rational Mech. Anal.* **109** (1990), 195–212.
- [6] H. FEDERER, *Geometric Measure Theory, Die Grundlehren der mathematischen Wissenschaften* **153**, Springer-Verlag, New York, 1969.
- [7] M. GHOMI, Solution to the shadow problem in 3-space, in *Minimal Surfaces, Geometric Analysis and Symplectic Geometry*, 2000, *Adv. Stud. Pure Math.* (To appear); preprint available at www.math.sc.edu/~ghomi.
- [8] M. GHOMI and B. SOLOMON, Skew loops and quadric surfaces, preprint available at www.math.sc.edu/~ghomi.
- [9] R. E. GREENE and S. G. KRANTZ, *Function Theory of One Complex Variable*, John Wiley & Sons Inc., New York, 1997.
- [10] M. GROMOV, *Partial Differential Relations*, Springer-Verlag, New York, 1986.
- [11] R. HOWARD, Mohammad Ghomi's solution to the shadow problem, Lecture notes, available at www.sc.edu/~howard.
- [12] N. H. KUIPER, Geometry in curvature theory, in *Tight and Taut Submanifolds* (Berkeley, CA, 1994), pp. 1–50, Cambridge Univ. Press, Cambridge, 1997.
- [13] J. MCCUAN, Personal e-mail, June 23, 1998.
- [14] J. MILNOR, *Morse Theory*, Based on lecture notes by M. Spivak and R. Wells, *Ann. of Math. Studies* **51**, Princeton Univ. Press, Princeton, NJ, 1963.
- [15] J. R. PORTER, A note on regular closed curves in E^3 , *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **18** (1970), 209–212.
- [16] B. SEGRE, Global differential properties of closed twisted curves, *Rend. Sem. Mat. Fis. Milano* **38** (1968), 256–263.
- [17] H. C. WENTE, Personal e-mail, January 9, 1999.
- [18] Y.-Q. WU, Knots and links without parallel tangents. *Bull. London Math. Soc.*, to appear.

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